BOSTON UNIVERSITY

GRADUATE SCHOOL

Thesis

Solutions of Rational Integral Equations for Complex Roots

by

Earl George Boyd

(A.B., University of Maine, 1920)

submitted in partial fulfilment of the

requirements for the degree of

Master of Arts

~~1935~~

1936

INTRODUCTION

The purpose of this thesis is to study several methods for the complete solution of a rational integral equation when complex roots are involved, and to determine if possible the best method or methods for their solution. The only special case considered will be De Moivres' Quintic and equations of other degrees which may be solved in like manner.

I believe more attention will be given to this particular subject in the future by text book writers, since the complex unit is being used today in the practical sciences and without doubt will be more extensively used in this field in the future.

INDEX

SYNOPSIS of THEOREMS for FINDING REAL ROOTS of a RATIONAL INTEGRAL EQUATION.

An algebraic equation of the N^{th} degree may be written with all its terms transposed to the first member, thus:

$$X^n + A_1 X^{n-1} + A_2 X^{n-2} + \ldots + A_{n-1} X + A_n = 0;$$

and if all the coefficients and the absolute term are real numbers, this is commonly called a numerical equation. The first member may be denoted for brevity by F (X) and the equation itself by F (X) = 0.

The following principles of the theory of algebraic equations with real coefficients, deduced in text-books on algebra, are here recapitulated for convenience of reference:

(1) If X_1 is a root of the equation, F (X) is divisible by $(X-X_1)$, and conversely, if F (X) is divisible by $X-X_1$, then X_1 is a root of the equation.

(2) An equation of the N^{th} degree has N roots and no more.

(3) If $X_1, X_2, \ldots X_n$ are the roots of the equation, then the product $(X-X_1)(X-X_2)\ldots(X-X_n)$ is equal to F (X).

(4) The sum of the roots is equal to $--A_1$; the sum of the products of the roots, taken two in a set, is equal to $+ A_2$; the sum of the products of the roots, taken three

in a set, is equal to $-A_3$; and so on. The product of all the roots is equal to $-A_n$ when N is odd, and to $+ A_n$ when N is even.

(5) The equation $F(X) = 0$ may be reduced to an equation lacking its second term by substituting $Y - A_1/N$ for X.

(6) If an equation has imaginary roots, they occur in pairs of the form $P \pm QI$ where I represents $\sqrt{-1}$.

(7) An equation of odd degree, has at least one real root whose sign is opposite to that of A_n.

(8) An equation of even degree, having A_n negative, has at least two real roots, one being positive and the other negative.

(9) A complete equation cannot have more positive roots than variations in the signs of its terms, nor more negative roots than permanences in signs. If all roots be real, there are as many positive roots as variations, and as many negative roots as permanences.

(10) In an incomplete equation, if an even number of terms, say 2m, are lacking between two other terms, then it has at least 2m imaginary roots; if an odd number of terms, say $2m + 1$, are lacking between two other terms, then it has at least either $2m + 2$ or 2m imaginary roots, according as the two terms have like or unlike signs.

(11) Sturm's theorem gives the number of real roots, provided that they are unequal, as also the number of real roots lying between two assumed values of X.

(12) If A_r is the greatest negative coefficient, and if A_s is the greatest negative coefficient after X is changed into --X, then all real roots lie between the limits $A_r \neq 1$ and $-(A_s \neq 1)$.

A BRIEF OUTLINE of the ROOT SQUARING PROCESS

The principle is to derive an equation whose roots are powers of the roots of the given equation. The power being large enough so that the roots are widely separated. This is done by finding the equation whose roots are the squares of the roots of the original equation and then repeating the process with this new derived equation. Repeating this process as many times as is necessary to give the desired power to separate the roots. We may do this in the following way.

Let the given equation be

(I) $F(X) \equiv A_0X^n + A_1 X^{n-1} + A_2 X^{n-2} + ---- + A_{n-1}X + A_n = 0$

and the roots be R_1, R_2, R_3, $----R_n$. Then (I) may be written

(II) $F(X) \equiv A_0 (X-R_1)(X-R_2)(X-R_3)----(X-R_n)$

now multiplying (II) by the function

$(-1)^n F(-X) \equiv (-1)^n A_0 (-X-R_1)(-X-R_2)(-X-R_3)----(-X-R_n)$

and we have

(III) $(-1)^n F(-X)\, F(X) \equiv A_0^2 (X^2-R_1^2)(X^2-R_2^2)(X^2-R_3^2)----(X^2-R_n^2)$

Let $X^2 = Z$. Then (III) becomes

(IV) $U(Z) \equiv A_0^2 (Z-R_1^2)(Z-R_2^2)(Z-R_3^2)----(Z-R_n^2)$

now the roots of equation (IV) are R_1^2, R_2^2, R_3^2----R_n^2, and are therefore the squares of the roots of equation (I).

This multiplication may be done by a fairly easy method as follows

consider the fifth degree equation

(V) $F(X) \equiv A_0 X^5 + A_1 X^4 + A_2 X^3 + A_3 X^2 + A_4 X + A_5$

Then

(VI) $(-1)^5 F(-X) \equiv A_0 X^5 - A_1 X^4 + A_2 X^3 - A_3 X^2 + A_4 X - A_5$

By actually multiplying (V) by VI we have

$(-1)^5 F(-X) F(X) \equiv A_0^2 X^{10} - A_1^2 X^8 + 2A_0 A_2 X^8 + A_2^2 X^6 - 2A_1 A_3 X^6 + 2A_0 A_4 X^6 - A_3^2 X^4 + 2A_2 A_4 X^4 - 2A_1 A_5 X^4 + A_4^2 X^2 - 2A_3 A_5 X^2 - A_5^2$

Collecting coefficients

(v) $A_0 X^5 + A_1 X^4 + A_2 X^3 + A_3 X^2 + A_4 X + A_5$

A_0^2	X^{10}	$-A_1^2$	X^8	$+A_2^2$	X^6	$-A_3^2$	X^4	$+A_4^2$	X^2	$-A_5^2$
		$2A_0 A_2$		$-2A_1 A_3$		$+2A_2A_4$		$-2A_3A_5$		
				$+2A_0 A_4$		$-2A_1 A_5$				

If we had started with an equation of even degree and performed the multiplication in like manner we would have found that the law for the formation of the coefficient would have been the same. The coefficient may

therefore be written down according to the following rule:

1. The numbers in the top row are the squares of the coefficients directly above them, with alternating signs--the second, fourth, sixth, etc. squared numbers being negative.

2. The quantities directly under these squared numbers are the doubled products of the coefficients equally removed from the one directly overhead, the first being twice the product of the two coefficients adjacent to the one overhead, the second the doubled product of the next two equally removed coefficients, etc.

3. The signs of the doubled products are changed alternately in going along the rows and also in going down the columns, the sign of the first doubled product in each row not being changed.

Let us assume the equation to be

$$B_0 (X^m)^n + B_1 (X^m)^{n-1} + --- + B_{n-1} (X^m) + B_n = 0$$

after the process has been repeated enough times to widely separate the roots.

Using the following relationships between coefficients and roots

$$\frac{B_1}{B_0} = (R_1^m + R_2^m + R_3^m + --- R_n^m)$$

$$\frac{B_2}{B_0} = R_1^m R_2^m + R_1^m R_3^m + \text{---} + R_1^m R_n^m + R_2^m R_3^m + R_2^m R_4^m$$

$$+ \text{---} R_2^m R_n^n + \text{---} + R_{n-1}^m R_n^m$$

$$\frac{B_3}{B_0} = -(R_1^m R_2^m R_3^m + R_1^m R_2^m R_4^m + \text{------}$$

$$\frac{B_n}{B_0} = (-1)^n R_1^m R_2^m \text{----} R_n^m$$

Where $R_1 > R_2 > R_3 > \text{----} > R_n$

Now if the roots are widely separated the first terms in the right hand members of these equations predominate. That is the others are negligible in comparison--assuming the M^{th} powers are such as to give us the required decimal accuracy.

then

$$\frac{B_1}{B_0} = -R_1^m \text{ (approximately)}$$

$$\frac{B_2}{B_0} = R_1^m R_2^m \quad \text{"}$$

$$\frac{B_3}{B_0} = -R_1^m R_2^m R_3^m \quad \text{"}$$

$$\frac{B_n}{B_0} = (-1)^m R_1^m R_2^m \text{----} R_n^m$$

Dividing each equation after the first by the preceding equation, we have

$$\frac{B_2}{B_1} = -R_2^m, \quad \frac{B_3}{B_2} = -R_3^m, \quad \frac{B_n}{B_{N-1}} = -R_n^m.$$

Now to find the roots of the equation all we need to do is solve the binomial equations.

$$B_1 + B_0 R_1^m = 0$$

$$B_2 + B_1 R_2^m = 0$$

$$B_3 + B_2 R_3^m = 0$$

$$B_n + B_{n-1} X_n^m = 0$$

The solution may be done conveniently by logarithms.

When the roots are all real. After the first two or three transformations the signs of the coefficients of the terms in the transformed equations alternate in sign and remain with the same sign for all the remaining transformations. If the roots are either positive or negative when they are squared they all become positive and the equation with all positive roots alternate in sign.

Also the doubled products disappear for the required number of decimals. As the transformations are made the coefficients become very large and it is only necessary to keep as many digits accurately as we desire accuracy in our roots. The discussion for the disappearance of the doubled products may be read on page 12. In this discussion it will be shown that if the doubled products do not disappear for a particular row that it will be due to the presence of a pair of complex roots.

THE ROOT SQUARING METHOD WHERE ONE PAIR of COMPLEX ROOTS OCCUR

When the equation has a pair of complex roots it cannot be expressed as a product of linear factors with real coefficients. It may be expressed as a product of linear equations and a single quadratic, the quadratic having the pair of complex roots for its roots. The root squaring method therefore breaks it up into linear factors and a quadratic factor.

The complex root may be noted in two ways:

* (1) the doubled products do not disappear from one column and (See page 12 - 2^{nd} term of quadratic fragment)

(2) the signs in this particular column fluctuate. (See note on page 8)

$(a \neq b,\ a \neq 0)$

Assume we have x_1, $a + b\,i$, $a - b\,i$, x_2 and x_3 for the roots of an equation with the order of magnitude being $|x_1| > |r| > |x_2| > |x_3|$ where "r" is the modulus of the complex numbers which are roots of the equation that is

$a \pm b\,i = r\,(\cos\theta \pm i \sin\theta)$.

now $e^{i\theta} = \cos\theta + i \sin\theta$

$r\,e^{i\theta} = r\,(\cos\theta + i\sin\theta)$

*The Column is dependent on the magnitude of the modulus of the Complex roots compared to the real roots.

and $e^{-i\theta} = \cos(-\theta) + i \sin(-\theta)$

$= \cos\theta - i \sin\theta$

$r\, e^{-i\theta} = r(\cos\theta - i\sin\theta)$

Then the equation having these roots is

I $\quad (x - x_1)(x - r e^{i\theta})(x - r e^{-i\theta})(x - x_2)(x - x_3) = 0.$

and the equation having for its roots the m^{th} powers of the roots of equation (I) is

II $\quad (y - x_1^m)(y - r^m e^{im\theta})(y - r^m e^{-im\theta})(y - x_2^m)$

$(y - x_3^m) = 0.$

where $y = x^m$, Multiplying we have

$y^5 - (x_1^m + x_2^m + x_3^m + r\, e^{im\theta} + r\, e^{-im\theta})\, y^4$

$+ (x_1^m x_2^m + x_1^m x_3^m + \text{---} + x_2^m x_3^m + \text{---} + r^m e^{im\theta} r^m e^{-im\theta}$

$+ \text{----})\, y^3$

$- (x_1^m x_2^m x_3^m + x_1^m x_2^m r^m e^{im\theta} + \text{---} + x_2^m r^m e^{im\theta} r^m e^{-im\theta}$

$+ \text{---})\, y^2$

$+ (x_1^m x_2^m x_3^m r^m e^{im\theta} + x_1^m x_2^m x_3^m e^{-im\theta} + x_2^m x_3^m r^m e^{-im\theta})\, y$

$- x_1^m x_2^m x_3^m r^m e^{im\theta} r^m e^{-im\theta}$

Since $|x_1| > |r| > |x_2| > |x_3|$ the term of greatest modulus is the predominating part of the coefficient of y^4, x_1^m, for y^3 it is $x_1^m r^m e^{im\theta} + x_1 r^m e^{-im\theta} =$

$$x_1^m\, r^m\, (e^{im\theta} + e^{-im\theta}) =$$

$$x_1^m\, r^m\, (\cos m\cdot\theta + i \sin m\cdot\theta + \cos m\cdot\theta - i \sin m\cdot\theta) =$$

$$x_1^m\, r^m\, (2 \cos m\cdot\theta)$$

For y^2, $x_1^m\, (r^m\, e^{im\theta})\, (r^m\, e^{-im\theta}) = x_1^m\, r^{2m}$

For y, $x_1^m\, r^{2m}\, x_2^m$

Then we have for the transformed equation whose roots are approximately the m^{th} powers of the roots of the initial equation.

$$\text{III} \quad y^5 - x_1^m\, y^4 + 2x_1^m\, r^m \cos(m\cdot\theta) y^3 - x_1^m\, r^{2m}\, y^2 + x_1^m\, r^{2m}\, x_2^m\, y - x_1^m\, r^{2m}\, x_2^m\, x_3^m = 0$$

Definition: The word fragment is to indicate any group of terms taking in sequence when the equation is arranged according to the descending powers of the unknown.

Now taking the following fragments from (III) and solving as equations:

(a) $y^5 - x_1^m\, y^4 = 0 \quad y^4\, (y - x_1^m) = 0 \quad y = x_1^m$

(b) $- x_1^m\, y^4 + 2x_1^m\, r^m \cos(m\cdot\theta)\, y^3 - x_1^m\, r^{2m}\, y^2 = 0.$

$$- x_1^m\, y^2\, (y^2 - 2r^m \cos(m\cdot\theta)\, y + r^{2m}) = 0.$$

$$y^2 - r^m\left[\{\cos m\theta + i \sin(m\cdot\theta)\} + \{\cos(m\cdot\theta) - i \sin m\cdot\theta\}\right]y - r^{2m} = 0$$

$$y^2 - r^m\, (e^{im\theta} + e^{-im\theta})\, y - r^{2m} = 0.$$

$$(y - r^m\, e^{im\theta})\, (y - r^m\, e^{-im\theta}) = 0. \quad y = r^m\, e^{im\theta}, \quad y = r^m\, e^{-im\theta}$$

(c) $-x_1^m\, r^{2m}\, y^2 + x_1^m\, r^{2m}\, x_2^m\, y = 0$

$$- x_1^m\, r^{2m}\, y\, (y - x_2^m) = 0 \qquad y = x_2^m$$

(d) $x_1^m\, r^{2m}\, x_2^m\, y - x_1^m\, r^{2m}\, x_2^m\, x_3^m = 0$

$$x_1^m\, r^{2m}\, x_2^m\, (y - x_3^m) = 0 \qquad y = x_3^m$$

From these equations we do obtain the roots with which we started on page 9.

Applying the root squaring process to (III) once more.

	y^5	y^4	y^3	y^2
m^0	1	$-x_1^m$	$2x_1^m r^m \cos m\theta$	$-x_1^m r^{2m}$
	1	$-x_1^{2m}$	$4x_1^{2m} r^{2m} \cos^2 m\theta$	$-x_1^{2m} r^{4m}$
		$4x_1^m r^m \cos m\theta$	$-2x_1^{2m} r^{2m}$	$4x_1^{2m} r^{3m} x_2^m \cos m\theta$
			$2x_1^m r^{2m} x_2^m$	$2x_1^{2m} r^{2m} x_2^m x_3^m$
$(2m)^0$	1	$-x_1^{2m}$	$4x_1^{2m} r^{2m} \cos^2 m\theta$	$-x_1^{2m} r^{4m}$
			$-2x_1^{2m} r^{2m}$	

y	y^0
$x_1^m r^{2m} x_2^m$	$-x_1^m r^{2m} x_2^m x_3^m$
$x_1^{2m} r^{4m} x_2^{2m}$	$-x_1^{2m} r^{4m} x_2^{2m} x_3^{2m}$
$2x_1^{2m} r^{4m} x_2^m x_3^m$	
$x_1^{2m} r^{4m} x_2^{2m}$	$-x_1^{2m} x^{4m} x_2^{2m} x_3^{2m}$

If we divide the doubled term in each column by the squared term at the top we shall see that these quotients are negligible except one in the first row, column 3, These divisions will indicate the dominant terms of each coefficient.

Taking the coefficient of y^3 as an illustration

$$\frac{2x_1^m\ r^{2m}\ x_2^m}{4x_1^{2m}\ r^{2m}\ \cos^2 m\cdot\theta} = \frac{x_2^m}{2x_1^m\ \cos^2\ m\cdot\theta}$$

which is negligible since the roots are widely separated and

$|x_1| > |r| > |x_2| > |x_3|$.

If $m\ \theta$ is near 90° by repeating the process several times $m\ \theta$ will not be near enough 90° to affect the quotient from being negligible. $m\ \theta$ cannot be 90° since $a \neq b$ in the complex roots.

Now $2\cos^2\theta - 1 = \cos 2\cdot\theta$

then the coefficient of y^3 may be written

$$4x_1^{2m}\ r^{2m}\ \cos^2\ m\cdot\theta - 2x_1^{2m}\ r^{2m}$$

$$2x_1^{2m}\ r^{2m}\ (2\cos^2\ m\cdot\theta - 1)$$

$$2x_1^{2m}\ r^{2m}\ \cos(2\ m\cdot\theta)$$

Then the last transformed equation whose roots are of the $2m^{th}$ power are:

$$\text{(IV)}\quad y^5 - x_1^{2m}\ y^4 + 2x_1^{2m}\ r^{2m}\ \cos(2\ m\cdot\theta)\,y^3 - x_1^{2m}\ r^{4m}\ y^2 + x_1^{2m}\ r^{4m} + x_1^{2m}\ r^{4m}\ x_2^{2m}\ y - x_1^{2m}\ r^{4m}\ x_2^{2m}\ x_3^{2m} = 0$$

On comparing the coefficients of equation (IV) with those of equation (III) we see that the application of the root squaring method doubles the amplitude of the complex roots. As the amplitudes are doubled the cosines frequently change signs. This is the reason for the fluctuation of the signs in one of the coefficients when complex roots are present.

RELATIONS between the COEFFICIENTS of an ALGEBRAIC EQUATION AND the RECIPROCALS of its ROOTS.

In the equation

$$a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + --- a_{n-2} x^2 + a_{n-1} x + a_n = 0.$$

Let $x = 1/y$ and clearing of fractions we have

$$a_n y^n + a_{n-1} y^{n-1} + a_{n-2} y^{n-2} + --- a_2 y^2 + a_1 y + a_0 = 0.$$

now from the relationship between roots and coefficients we have

$$\frac{a_{n-1}}{a_n} = -(y_1 + y_2 + y_3 + ---y_n)$$

$$\frac{a_{n-2}}{a_n} = y_1 y_2 + y_1 y_3 + y_1 y_4 + --+ y_2 y_3 + y_2 y_4 +---$$

$$\frac{a_0}{a_n} = (-1)^n y_1 y_2 y_3 ---y_n$$

and since $y = 1/x$

$$\frac{a_{n-1}}{a_n} = -(1/x_1 + 1/x_2 + 1/x_3 + --- + 1/x_n$$

$$\frac{a_{n-2}}{a_n} = 1/x_1 x_2 + 1/x_1 x_3 + 1/x_1 x_4 +---+ 1/x_2 x_3 + 1/x_2 x_4 +--$$

These relationships between the coefficients and reciprocals of the roots will assist us in avoiding ambiquities of signs in the computation of the complex roots.

Example: $x^4 - 6x^3 + 11x^2 + 2x - 28 = 0$

The small n and c will denote no change in sign or a change in sign.

	x^4	x^3	x^2	x^1	x^0
Eq.	1	-6	11	2	-28
	1	-36	121	-4	784
		22 n	24 c	-616 n	
			-56 n		
P^2	1	$-1.4(10)$	$8.9(10)$	$-6.2(10)^2$	$7.84(10)^2$
	1	$-1.96(10)^2$	$7.921(10)^3$	$-3.844(10)^5$	$6.147(10)^5$
		1.78 n	-17.36 c	1.396 n	
			1.568 n		
P^4	1	$-.18(10)^2$	$-7.871(10)^3$	$-2.448(10)^5$	$6.147(10)^5$
	1	$-.032.10^4$	$6.195.10^7$	$-5.992.10^{10}$	$3.779.10^{11}$
		- 1.574 n	-.881	-.968 n	
			.123 n		
P^8	1	$-1.606(10)^4$	$5.437(10)^7$	$-6.960(10)^{10}$	$3.779(10)^{11}$
	1	$-2.580(10)^8$	$2.956(10)^{15}$	$-4.843(10)^{21}$	$1.428(10)^{23}$
		1.087 n	-2.236 c	.041 n	
			.001 n		
P^{16}	1	$-1.493(10)^8$	$-721(10)^{15}$	$-4.802(10)^{21}$	$1.428(10)^{23}$

	1	$-2.228(10)^{16}$ $.144$ n	$.520(10)^{30}$ -1.434 c * n	$-2.305(10)^{43}$ * n	$2.038(10)^{46}$
P^{32}	1	$-2.084(10)^{16}$	$-.914(10)^{30}$	$-2.305(10)^{43}$	$2.038(10)^{46}$
	1	$-4.342(10)^{32}$ $-\ .018$ n	$.835(10)^{60}$ $-.960$ c	$-5.314(10)^{86}$ * n	$4.153(10)^{92}$
P^{64}	1	$-4.360(10)^{32}$	$-.125(10)^{60}$	$-5.314(10)^{86}$	$4.153(10)^{92}$

We see that another application of the root squaring method would not further separate the roots.

The given equation has now been broken up into two linear and one quadratic fragment, taking the first linear fragment and computing the root by logarithms.

$$x_1^{64} = 4.360.10^{32} \qquad (b_0 x^m + b_1 = 0)$$

$$64 \log x_1 = \log 4.360 + 32 \log 10$$

$$64 \log x_1 = .639486 + 32$$

$$\log x_1 = .509992$$

$$x_1 = 3.2359$$

Solving the second linear fragment

$$5.314.10^{86} \; x_2^{64} = 4.153.10^{92}$$

$$5.314 \; x_2^{64} = 4.153.10^{6}$$

$$x_2^{64} = \frac{4.153.10^{6}}{5.314}$$

$$\log 4.153 = .618362$$
$$6 \log 10 = 6.$$
$$\text{Colog } 5.314 = 9.274578-10$$
$$64 \log x_2 = 15.892940-10$$
$$\log x_2 = .092077$$
$$x_2 = 1.2362$$

On substituting -1.2 we find it nearly satisfies the equation. On substituting 3.2 we find it nearly satisfies the equation. Then the real roots are

$$x_1 = 3.2359$$
$$x_2 = -1.2362$$

The modulus for the pair of complex roots is found from the equation

$$-4.360 \cdot 10^{32} y^2 - .125 \cdot 10^{60} y - 5.314 \cdot 10^{86} = 0$$

when $y = x^{64}$

Dividing by the coefficient of y^2 we have

$$y^2 + \frac{.125 \cdot 10^{28}}{4.360} y + \frac{5.314 \cdot 10^{54}}{4.360} = 0$$

Now from the footnote on page 11 we see that the absolute term in the quadratic is equal to the square of the modulus of the complex roots. Let R be this modulus

then $(R = r^{64})$

$$R^2 = \frac{5.314 \cdot 10^{54}}{4.360}$$

Solving by logarithms

$$\log 5.314 = .725422$$

$$54 \log 10 = 54.000000$$

$$\text{colog } 4.360 = 9.360514-10$$

$$2 \log R = 54.085936$$

$$\log R\ (r^{64}) = 27.042968$$

$$\log r = .422547$$

$$r = 2.6457$$

Let the pair of complex roots be $u \neq Vi$, $u - Vi$.

Now since the sum of the roots of the equation is (-6), we have

$$x_1 \neq x_2 \neq 2u = -6$$

$$2u = \neq 6 - 3.2359 \neq 1.2362$$

$$u = \neq 1.9999 = \neq 2 \text{ app.}$$

Now $u^2 \neq v^2 = r^2$

$v^2 = r^2 - u^2$ Substituting values of u and r

$$v = \sqrt{3}$$

The complex roots are therefore

$$2 \neq \sqrt{3}i,\ 2 - \sqrt{3}i$$

We may check by finding the product of the roots and comparing with the constant term

$x_1\ x_2\ r^2 = 28.00153$ which is as near as we could expect.

TWO PAIR of COMPLEX ROOTS

If two pairs of complex roots are present the fluctuations in signs will occur in two columns. Let the roots of an equation be, x_1, $r_1 e^{i\theta_1}$, $r_1 e^{-i\theta_1}$, $r_2 e^{i\theta_2}$, $r_2 e^{-i\theta_2}$ and the order of magnitude to be

$|r_1| > |x_1| > |r_2|$. Then the equation is

$$(x - r_1 e^{i\theta_1})(x - r_1 e^{-i\theta_1})(x - x_1)(x - r_2 e^{i\theta_2})$$
$$(x - r_2 e^{-i\theta_2}) = 0.$$

and the equation whose roots are the m^{th} powers of the roots of this equation will be

$$(y - r_1^m e^{im\theta})(y - r_1^m e^{-im\theta})(y - x_1^m)(y - r_2^m e^{im\theta_2})$$
$$(y - r_2^m e^{-im\theta_2}) = 0, \text{ where } y = x^m$$

Expanding and taking the leading terms we have

$$y^5 - 2r_1^m \cos m\theta_1\, y^4 + r_1^{2m} y^3 - x_1^m r_1^{2m} y^2$$
$$+ 2x_1^m r_1^{2m} r_2^m \cos m\theta_2\, y - x_1^m r_1^{2m} r_2^{2m} = 0.$$

The linear and quadratic fragments being

$$y^5 - 2r_1^m \cos m\theta_1\, y^4 + r_1^{2m} y^3 = 0$$
$$r_1^{2m} y^3 - x_1^m r_1^{2m} y^2 = 0$$
$$- x_1^m r_1^{2m} y^2 + 2x_1^m r_1^{2m} r_2^m \cos m\theta_2\, y - x_1^m r_1^{2m} r_2^{2m} = 0.$$

The roots of these fragments being the roots that we started with. The middle terms of the quadratic fragments are the terms in which the signs fluctuate.

$$x^6+x^5+6x^4-20x^3+51x^2-53x-130$$

	x^6	x^5	x^4	x^3	x^2	x^1	x^0
Given Equation	+1	+1	+6	-20	+51	-53	-130
	+1	$-.1\cdot10$ $+1.2$ n	$+.36\cdot10^{2}$ $+.40$ c $+1.02$ n	$-4.00\cdot10^{2}$ $+6.12$ n $+1.06$ c -2.60 n	$+2.601\cdot10^{3}$ -2.120 c -1.560 n	$-.281\cdot10^{4}$ -1.326 n	$+1.69\cdot10^{4}$
2nd Power	+1	$+1.1\cdot10$	$+1.78\cdot10^{2}$	? $.58\cdot10^{2}$	$-1.079\cdot10^{3}$	$-1.607\cdot10^{4}$	$+1.69\cdot10^{4}$
	+1	$-1.21\cdot10^{2}$ $+3.56$ n	$+3.168\cdot10^{4}$ $-.128$ c $-.216$ n	$-.034\cdot10^{5}$ -3.841 n $+3.535$ c $+338$ n	$+1.164\cdot10^{6}$ $+1.864$ c $+6.016$ n	$-2.582\cdot10^{8}$ $-.365$ n	$+2.856\cdot10^{8}$
4th Power	+1	$+2.35\cdot10^{2}$	$+2.824\cdot10^{4}$	$-.002\cdot10^{5}$	$+9.044\cdot10^{6}$	$-2.947\cdot10^{8}$	$+2.856\cdot10^{8}$
	+1	$-5.523\cdot10^{4}$ $+5.648$ n	$+7.975\cdot10^{8}$ $+.001$ c $+.181$ n	$-.000\cdot10^{11}$ $+5.106$ n $+1.680$ c $+.006$ n	$+8.179\cdot10^{13}$ $-.012$ c $+1.613$ n	$-8.685\cdot10^{16}$ $+.517$ n	$+8.157\cdot10^{16}$
8th Power	+1	$+.125\cdot10^{4}$	$+8.157\cdot10^{8}$	$+6.792\cdot10^{11}$	$+9.780\cdot10^{13}$	$-8.168\cdot10^{16}$	$+8.157\cdot10^{16}$
	+1	$-.016\cdot10^{9}$ $+1.631$ n	$+6.654\cdot10^{17}$ $-.017$ c $+.002$ n	$-4.613\cdot10^{23}$ $+1.596$ n $+.002$ c $+.000$ n	$+.096\cdot10^{29}$ $+1.109$ c $+.001$ n	$-6.672\cdot10^{33}$ $+.016$ n	$+6.654\cdot10^{33}$
16th Power	+1	$+1.615\cdot10^{9}$	$+6.639\cdot10^{17}$	$-3.015\cdot10^{23}$	$+1.206\cdot10^{29}$	$-6.656\cdot10^{33}$	$+6.654\cdot10^{33}$
	+1	$-2.608\cdot10^{18}$ $+1.328$ n	$+4.408\cdot10^{35}$ $+.010$ c *	$-.909\cdot10^{47}$ $+1.601$ n *	$+1.454\cdot10^{58}$ $-.401$ c *	$-4.430\cdot10^{67}$ * n	$+4.428\cdot10^{67}$
32nd Power	+1	$-1.280\cdot10^{18}$	$+4.418\cdot10^{35}$	$+.692\cdot10^{47}$	$+1.053\cdot10^{58}$	$-4.430\cdot10^{67}$	$+4.428\cdot10^{67}$
	+1	$-1.638\cdot10^{36}$ $+.884$ n	$+1.952\cdot10^{71}$ * c	$-.479\cdot10^{94}$ $+.930$ n	$+1.109\cdot10^{116}$ $+.061$ c	$-1.962\cdot10^{135}$	$+1.961\cdot10^{135}$
64th Power	+1	$-.754\cdot10^{36}$	$+1.952\cdot10^{71}$	$+.431\cdot10^{94}$	$+1.170\cdot10^{116}$	$-1.962\cdot10^{135}$	$+1.961\cdot10^{135}$
	+1	$-5.685\cdot10^{71}$ $+3.904$ n	$+3.810\cdot10^{142}$ *	$-1.858\cdot10^{187}$ $+4.567$ n	$+1.369\cdot10^{232}$ $+.002$ c	$-3.849\cdot10^{270}$	$+3.846\cdot10^{270}$
128th Power	+1	$-1.781\cdot10^{71}$	$+3.810\cdot10^{142}$	$+2.709\cdot10^{187}$	$+1.371\cdot10^{232}$	$-3.849\cdot10^{270}$	$+3.846\cdot10^{270}$

With this equation the fluctuations in signs will occur in the coefficients of y^4 and y. Other results hold as in case I for the same reasons.

In the illustrative example on page (21).

We have broken the equation up into two linear and two quadratic fragments. The first real root is found from the linear fragment.

$$1.371\cdot10^{232}\ y = 3.849\cdot10^{270} \quad (y = x_1^{128})$$

$$x_1^{128} = \frac{3.849\cdot10^{38}}{1.371}$$

Then by logarithms $x_1 = 1.997$ (app. 2)

The second real root is found from the linear fragment

$$3.849\cdot10^{270}\ y = 3.846\cdot10^{270} \quad y = x_2^{128}$$

$$x_2^{128} = 1 \text{ (app.)}$$

$$\log x_2 = 0$$

$$x_2 = 1$$

By substituting we find the signs to be $x_1 = 2$, $x_2 = -1$

The modulus of the first pair of complex roots is found from the quadratic fragment $y^2 - 1.781\cdot10^{71}y + 3.810\cdot10^{142} = 0$.

as under example for one pair of complex roots

$$(r_1^{128})^2 = 3.810\cdot10^{142}$$

Solving by logarithms we find

$$r_1 = 3.6055$$

The modulus of the second pair of complex roots is found from the quadratic fragment

$$3.810 \cdot 10^{142} y^2 + 2709 \cdot 10^{187} y + 1.371 \cdot 10^{232} = 0$$

$$y^2 + \frac{2.709 \cdot 10^{187}}{3.810 \cdot 10^{142}} y + \frac{1.371 \cdot 10^{232}}{3.810 \cdot 10^{142}} = 0$$

then $(r_2^{128})^2 = \dfrac{1.371 \cdot 10^{132}}{3.810 \cdot 10^{142}}$

by logarithms

$$r_2 = 2.238$$

We will now let the complex roots be

$$a_1 + b_1 i, \; a_1 - b_1 i, \; a_2 - b_2 i, \; a_2 + b_2 i,$$

Then the sum of the roots is

$$x_1 + x_2 + 2a_1 + 2a_2 = -1 \qquad (x_1 + x_2 = -1)$$

(I) $a_1 + a_2 = -1$

Now applying the theorem on page 15 for the reciprocals of the roots

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{a_1 + b_1 i} + \frac{1}{a_1 - b_1 i} + \frac{1}{a_2 - b_2 i} + \frac{1}{a_2 + b_2 i} = -\frac{53}{130}$$

rationatizing the denominators and substituting

$a_1^2 + b_1^2 = r_1^2$ and $a_2^2 + b_2^2 = r_2^2$

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{2a_1}{r_1^2} + \frac{2a_2}{r_2^2} = -\frac{53}{130}$$

now the values substituted and dividing by 2

$\frac{1}{x_1} = .5, \ \frac{1}{x_2} = -1, \ \frac{1}{r_1^2} = .0769, \ \frac{1}{r_2^2} = .1997$ (.2 app)

gives

$$.0769\, a_1 + .2\, a_2 = .0462$$

now $\quad a_1 + a_2 = -1 \quad$ (I)

Solving simultaneously

$$a_1 = -2, \quad a_2 = 1$$

now $a_1^2 + b_1^2 = r_1^2$ and $a_2^2 + b_2^2 = r_2^2$

substituting and solving for b_1 and b_2

$$b_1 = 3, \quad b_2 = 2$$

Then the two pair of complex roots are

$$-2 \pm 3i \quad \text{and} \quad 1 \pm 2i$$

Now we may check from the relationship

$$x_1\, x_2\, r_1^2\, r_2^2 = -130$$

$$(2)\,(-1)\,(13)\,(5) = -130$$

$$-130 = -130.$$

ONE PAIR of COINCIDENT COMPLEX ROOTS (Original)

Let a pair of coincident complex roots be $(r\,e^{i\theta})^2$, $(r\,e^{-i\theta})^2$ and the order of magnitude of the roots be $|r| > |x_1| > |x_2|$ of the equation $f(x) = 0$. That is

$$(x - r\,e^{i\theta})^2 (x - r\,e^{-i\theta})^2 (x - x_1)(x - x_2) = 0.$$

and the equation whose roots are the m^{th} powers of these roots is

$$(y - r^m e^{im\theta})^2 (y - r^m e^{-im\theta})^2 (y - x_1^m)(y - x_2^m) = 0.$$

On multiplying and taking the leading coefficients we have

$$y^6 - 4r^m \cos m\theta\, y^5 + 4r^{2m} y^4 - 4r^{3m} \cos m\theta\, y^3 + r^{4m} y^2$$
$$- x_1^m r^{4m} y + x_1^m x_2^m r^{4m} = 0.$$

The fragments are therefore

$$\text{I} \quad y^6 - 4r^m \cos m\theta\, y^5 + 4r^{2m} y^4 = 0$$
$$\text{II} \quad 4r^{2m} y^4 - 4r^{3m} \cos m\theta\, y^3 + r^{4m} y^2 = 0$$
$$r^{4m} y^2 - x_1 r^{4m} y = 0$$
$$-x_1^m r^{4m} + x_1^m x_2^m r^{4m} = 0$$

Finding the equation whose roots are the $2m^{th}$ powers of the given roots we get

$$y^6 - 8r^{2m} \cos 2m\theta\, y^5 + (18r^{4m} - 32\, r^{4m} \cos^3 m\theta)y^4$$
$$-8r^{6m} \cos 2m\theta\, y^3 + r^{8m} y^2 - x_1^{2m} r^{8m}\, y + x_1^{2m} x_2^{2m} \cdot r^{8m} = 0.$$

Now taking fragment I

$$y^6 - 4r^m \cos(m\cdot\theta)y^5 + 4r^{2m} y^4 = 0$$

$$y^4 (y^2 - 4r^m \cos(m\cdot\theta)y + 4r^{2m}) = 0$$

$$y^2 - (2r^m)\left[\{\cos m\cdot\theta + i \sin m\cdot\theta\} + \{\cos m\theta - i \sin m\theta\}\right] y + (2r^m)^2 = 0$$

$$y^2 - (2r^m)\ (e^{im\theta} + e^{-im\theta})\ y + (2r^m)^2 = 0$$

$$(y - 2r^m e^{im\theta})(y - 2r^m e^{-im\theta}) = 0$$

$$y = 2r^m e^{im\theta}, \quad y = 2r^m e^{-im\theta}$$

Then the constant term of the first quadratic fragment is $4r^{2m}$.

For the second quadratic fragment

$$4r^{2m} y^4 - 4r^{3m} \cos(m\cdot\theta)y^3 + r^{4m} y^2 = 0$$

$$r^{2m} y^2 (4y^2 - 4r^m \cos m\cdot\theta + r^{2m}) = 0$$

$$y^2 - r^m \cos m\cdot\theta + \frac{r^{2m}}{4} = 0.$$

$$y^2 - \frac{r^m}{2}\left[\{\cos m\cdot\theta + i \sin m\cdot\theta\} + \{\cos m\cdot\theta - i \sin m\cdot\theta\}\right] y + \left(\frac{r^m}{2}\right)^2 = 0.$$

$$y^2 - \frac{r^m}{2}\left(e^{im\theta} + e^{-im\theta}\right) + \left(\frac{r^m}{2}\right)^2 = 0$$

$$\left(y - \frac{r^m e^{im\theta}}{2}\right)\left(y - \frac{r^m e^{-im\theta}}{2}\right) = 0. \qquad Y = \frac{r^m e^{\pm im\theta}}{2}$$

Then the constant term of the second quadratic fragment is $\frac{r^{2m}}{4}$.

Now $4r^{2m}$ is the last term of fragment (I) and the first term of fragment (II). The roots of fragment (I) are twice the m^{th} powers of the original complex roots and the roots of fragment (II) are one half the m^{th} powers of the original complex roots. Therefore the roots of these fragments are

functionally related by the intermediate term ($4r^{2m}$) and therefore will not disappear as the root squaring process is continued.

In the transformed equation whose roots are the $2m^{th}$ powers of the original roots we see that this intermediate term does not disappear.

In the illustrative problem the coefficient between the coefficient fluctuating in sign indicates coincident roots. Solving under this supposition for the modulus "r" we find that the moduli are identical.

Of course it would be possible to solve for the multiple roots by finding the H. C. F. of $f(x) = 0$ and $f'(x) = 0$. I do not believe though that this test would be tried first if an equation was to be solved. If the root square method was tried first and there were coincident roots it would be a great deal easier to continue and find all roots by this method, provided all roots are desired.

	y^6	y^5	y^4	y^3
m^{th} Power	1	$-4r^{m}\cos m\theta$	$+4r^{2m}$	$-4r^{3m}\cos m\theta$
	$+1$	$-16r^{2m}\cos^2 m\theta$ $8r^{2m}$ n	$+16r^{4m}$ $-32r^{4m}\cos^2 m\theta$ c $2r^{4m}$ n	$-16r^{6m}\cos^2 m\theta$ $8r^{6m}$ n $-8r^{5m}x_1^{m}\cos m\theta$ c $2x_1^{2m}x_2^{2m}r^{8m}$ n
$2m^{th}$ Power	1	$-16r^{2m}\cos^2 m\theta$ $+8r^{2m}$	$18r^{4m}$ $-32r^{4m}\cos^2 m\theta$	$-16r^{6m}\cos^2 m\theta$ $8r^{6m}$

y^2	y^1	y^0
$+r^{4m}$	$-x_1^{m}r^{4m}$	$+x_1^{m}x_2^{m}r^{4m}$
$+r^{8m}$ $-8x_1^{m}r^{7m}\cos m\theta$ $+8x_1^{m}x_2^{m}r^{6m}$	$-x_1^{2m}r^{8m}$ $+2x_1^{m}x_2^{m}r^{8m}$ n	$+x_1^{2m}x_2^{2m}r^{8m}$
$+r^{8m}$	$-x_1^{2m}r^{8m}$	$+x_1^{2m}x_2^{2m}r^{8m}$

$$4x^6+4x^4+36x^3-39x^2+70x-25=0$$

$$x^6+x^4+9x^3-9.75x^2+17.5x-6.25=0$$

	x^6	x^5	x^4	x^3	x^2	x^1	x^0
Given Equation	+1	+0	+1	+9	-9.75	+17.5	-6.25
	+1	-0 +2 n	+ .10·10 +0 c -1.95 n	-8.10·10 -1.95 n +0 c -1.25 n	+ .951·10^2 -3.150 c - .125 n	-3.063·10^2 +1.219 n	+3.906·10
2nd Power	+1	+2	-1.85·10	-1.13·10^2	-2.324·10^2	-1.844·10^2	+3.906·10
	+1	- .4·10 - 3.7 n	+3.423·10^2 +4.52 c -4.648 n	-1.277·10^4 + .860 n + .074 c + .008 n	+5.401·10^4 -4.167 c - .145 n	-3.400·10^4 -1.816 n	+1.526·10^3
4th Power	+1	-4.10·10	+3.295·10^2	- .335·10^4	+1.089·10^4	-5.216·10^4	+1.526·10^3
	+1	-1.681·10^3 + .659 n	+1.086·10^5 -2.747 c + .218 n	-1.222·10^7 + .717 n - .428 c + .000 n	+1.186·10^8 -3.495 c + .010 n	-2.721·10^9 + .033 n	+2.329·10^6
8th Power	+1	-1.022·10^3	-1.443·10^5	- .833·10^7	-2.299·10^8	-2.688·10^9	+2.329·10^6
	+1	-1.044·10^6 - .289 n	+2.082·10^{10} -1.703 c - .046 n	-6.939·10^{13} +6.634 n - .549 c	+5.285·10^{16} -4.478 c - * n	-7.225·10^{18} - .001 n	+5.424·10^{12}
16th Power	+1	-1.333·10^6	+3.330·10^9	-8.540·10^{12}	+8.070·10^{15}	-7.224·10^{18}	+5.424·10^{12}
	+1	-1.777·10^{12} + .007 n	+1.089·10^{19} -2.277 c + .002 n	-7.293·10^{25} +5.374 n -1.926 c	+6.512·10^{31} -12.339 c * n	-5.219·10^{37} * n	+2.942·10^{25}
32nd Power	+1	-1.770·10^{12}	-1.186·10^{19}	-3.845·10^{25}	-5.827·10^{31}	-5.219·10^{37}	+2.942·10^{25}
	+1	-3.133·10^{24} * n	+1.407·10^{38} -1.361 c	-1.478·10^{51} +1.382 n - .185 c	+3.395·10^{63} -4.013 c	-2.724·10^{75}	+8.655·10^{50}
64th Power	+1	-3.133·10^{24}	+ .046·10^{38}	- .281·10^{51}	- .618·10^{63}	-2.724·10^{75}	+8.655·10^{50}
	+1	-9.816·10^{48}	+ .021·10^{75} -1.761 c	-7.896·10^{100} - .448 n +1.707 c	+ .382·10^{126} -1.531 c	-7.420·10^{150}	+7.491·10^{101}
128th Power	+1	-9.816·10^{48}	-1.740·10^{75}	-6.637·10^{100}	-1.149·10^{126}	-7.420·10^{150}	+7.491·10^{101}
	+1	-9.635·10^{97}	+3.028·10^{150} +1.303 c	-4.405·10^{201} -3.998 n - .146 c	+1.320·10^{252} - .985 c	-5.506·10^{301}	+5.612·10^{203}
256th Power	+1	-9.635·10^{97}	+4.331·10^{150}	-8.549·10^{201}	+ .335·10^{252}	-5.506·10^{301}	+5.612·10^{203}

Taking the equation whose roots are the 32^{nd} powers of the roots of the original equations, since if we have coincident roots the roots are widely separated at this point.

$$1.770\cdot10^{12}\,y^2 + 1.186\cdot10^{19}\,y + 3.845\cdot10^{25} = 0$$

$$4R_1^2 = \frac{3.845\cdot10^{13}}{1.770} \quad \text{(1st quadractic fragment)}$$

$$R_1^2 = \frac{3.845\cdot10^{13}}{7.080} \qquad R_1 = r_1^{64}$$

$$\begin{aligned}
\log 3.845 &= .58490\\
13 \log 10 &= 13.\\
\text{colog } 7.080 &= \underline{9.14997-10}\\
64 \log r_1 &= 12.73487\\
\log r_1 &= .19898\\
r_1 &= 1.5812
\end{aligned}$$

$$+3.845\cdot10^{25}\,y^2 + 5.827\cdot10^{31}y + 5.219\cdot10^{37} = 0.$$

$$\frac{R_2}{4} = \frac{5.219\cdot10^{12}}{3.845} \quad \text{(2nd quadratic fragment)}$$

$$R_2 = \frac{20.876\cdot10^{12}}{3.845}$$

$$\begin{aligned}
\log 20.876 &= 1.31964\\
12 \log 10 &= 12.\\
\text{colog } 3.845 &= \underline{9.41510-10}\\
64 \log r_2 &= 12.73474\\
\log r_2 &= .19898\\
r_2 &= 1.5812
\end{aligned}$$

Now since $r_1 = r_2$ we have a pair of coincident roots.

Solving for the real roots:

$y^1 = 1.770 \cdot 10^{12}$

$x_1^{32} = 1.770 \cdot 10^{12}$

$\log 1.770 = .24797$

$12 \log 10 = 12.00000$

$32 \log x_1 = 12.24797$

$\log x_1 = .38275$

$x_1 = 2.414$

$5.219 \cdot 10^{37} \; y_2 = 2.942 \cdot 10^{25}$

$$x_2^{32} = \frac{2.942}{5.219 \cdot 10^{12}}$$

$\log 2.942 = .46864$

$\text{colog } 5.219 = 9.28241-10$

$\text{colog } 10^{12} = \underline{8.00000-20}$

$32 \log x_2 = 7.75105-20$

$\log x_2 = 9.61722-10$

$x_2 = .4142$

checking the real roots for the signs we find

$x_1 = -2.414, \quad x_2 = .4142$

Now taking the equation for the reciprocals of the roots and letting the complex roots be $a + bi$, $a - bi$

$$\frac{1}{x_1} + \frac{1}{x_2} + 2\left(\frac{1}{a + bi}\right) + 2\left(\frac{1}{a - bi}\right) = \frac{70}{25}$$

Rationalizing the denominators and placing $a^2 + b^2 = r^2$

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{4a}{r^2} = \frac{70}{25}$$

Now substituting the numerical values

$$\frac{1}{x_1} = .4143 \qquad \frac{1}{x_2} = 2.4143 \qquad \frac{1}{r^2} = .4$$

$$1.6\,a = .8$$

$$a = .5$$

Then the coincident roots are

$$.5 + 1.5812i, \quad .5 - 1.5812i$$

We may check by taking the negative sum of the roots which should be zero. We see that the sum is zero and the roots check.

SOLUTION BY MACLAURIN'S FORMULA

Given $f(y) = 0$ we may solve by expanding in MacLaurin's formula as follows:

If we place an x in all the terms of $f(y) = 0$ with the exception of the first and last, y becomes an implicit function of x. We may then obtain the successive derivatives of y in respect to x. These are evaluated for $x = 0$. Then by MacLaurin's formula the expansion of y in terms of x is known.

$$y = y_0 + \left(\frac{dy}{dx}\right)_0 x + \left(\frac{d^2y}{dx^2}\right)_0 \frac{x^2}{2!} + \left(\frac{d^3y}{dy^3}\right)_0 \frac{x^3}{3!} + - - -$$

y_0 are the values found for "y" when "x" is placed equal to zero and the resulting binomial equations are solved. The values of y_0 substituted in $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc., gives the values of $\left(\frac{dy}{dx}\right)_0$, $\left(\frac{d^2y}{dx^2}\right)_0$ etc., now letting $x = 1$ and substituting these values in the series will give the roots of $f(y) = 0$, providing these series are convergent.

If in placing the x in all terms but the first and last and expanding in MacLaurin's formula we find some of the series divergent, it becomes necessary to substitute the x in other terms including the first or last to get converging series. By varying these substitutions the several convergent series series giving the roots will be found.

In the illustrative examples we have a quartic which gives converging series for all of the roots when x is places in all of the terms but the first and last,

and a quartic where the series are divergent when x is placed in these terms. In this case by first placing x in the constant term we have two converging series giving two roots and again placing x in the leading term we have two other converging series giving the other two roots.

Solve:

$$y^4 - 7y^2 + 474y - 14.040 = 0$$

Introducing an x in all terms but the first and last

$$y^4 - 7y^2x + 474yx - 14,040 = 0.$$

$$\frac{dy}{dx} = \frac{7y^2 - 474y}{4y^3 - 14yx + 474x}$$

Let $N = 7y^2 - 474y$

$D = 4y^3 - 14yx - 474x$

$$\frac{d^2y}{dx^2} = \frac{D(14y - 474)\frac{dy}{dx} - N\left[(12y^2 - 14x)\frac{dy}{dx} - 14y + 474\right]}{D^2}$$

now if $x = 0$

y	=	10.89	-10.89	10.89i	-10.89i
$\frac{dy}{dx}$	=	-.84	- 1.16	1-.16i	1+ .16i
$\frac{d^2y}{dx^2}$	=	-.04	+ .08	0.00+.06i	0.00-.06i

Now substituting these values in MacLaurin's formula where $x = 1$

$$y = y_0 + \left(\frac{dy}{dx}\right)_0 x + \left(\frac{d^2y}{dx^2}\right) \frac{x^2}{2} + \cdots$$

and evaluating the four series we have

$y_1 = 10.03 \qquad y_3 = 1 + 10.76i$

$y_2 = -11.97 \qquad y_4 = 1 - 10.76i$

$y_1\, y_2\, y_3\, y_4 = 14.134$ (check)

Solve:

$y^4 + 13y^2 + 12 = 0$ *(Page 35)

$y^4 + 13y^2 + 12x = 0$

Placing x in the constant term

$$\frac{dy}{dx} = \frac{-6}{2y^3 + 13y}$$

$$\frac{d^2y}{dx^2} \quad \frac{6(6y^2 + 13)\,\frac{dy}{dx}}{(2y^3 + 13y)^2}$$

when $x = 0$

$y = 3.605i \qquad -3.605i$

$\frac{dy}{dx} = -.127i \qquad +.127i$

$\frac{d^2y}{dx^2} = -.032i \qquad +.032i$

Substituting in MacLaurin's series and evaluating

$y_1 = 3.462i, \quad y_2 = -3.462i$

Now placing x in the first term

$xy^4 + 13y^2 + 12 = 0.$

$$\frac{dy}{dx} = \frac{-y^4}{26y - 4xy^3}$$

$$\frac{d^2y}{dx^2} = \frac{-4y^3\,(26y - 4xy^3)\,\frac{dy}{dx} + y^4\,\left[(26 - 12xy^2)\,\frac{dy}{dx} - 4y^3\right]}{(26y - 4xy^3)^2}$$

when $x = 0.$

$$y = .9607\,i \qquad -.9607\,i$$

$$\frac{dy}{dx} = .0341\,i \qquad -.0341\,i$$

$$\frac{d^2y}{dx^2} = .0057\,i \qquad -.0057\,i$$

Substituting and evaluating

$$y_3 = .9977\,i, \qquad y_4 = -.9977\,i$$

$$y_1\,y_2\,y_3\,y_4 = 11.93 \text{ (check)}$$

* If we solve in the usual way

$$y^4 + 13x\,y^2 + 12 = 0$$

$$\frac{dy}{dx} = \frac{-13y^2}{4y^3 + 26x\,y}$$

$$\frac{d^2y}{dx} = \frac{(52y^4 - 676x\,y + 338y^2x)\frac{dy}{dx} + 338y^3}{(4y^3 + 26x\,y)^2}$$

when $x = 0$

$$y = \sqrt[4]{3} + i\sqrt[4]{3} = 1.32 + 1.32i$$

$$\frac{dy}{dx} = 2.28 - 2.28i$$

$$\frac{d^2y}{dx^2} = 162.9 + 0(i)$$

We see that the real part of the series gives a divergent series and therefore the entire series is divergent.

McClintock Method of Enlargement.

In the "American Journal of Mathematics," 1895, Vol. XVII, Pages 90-130, Emory McClintock has solutions for a number of equations for all roots by this method. The theory of the method was written by him in a previous paper. The writer could not find a reference to this work. In any event a complete discussion of this method would not be included within the scope of this thesis, as a close study of the Journal shows that if the original papers could be found there would be enough material for a complete thesis in itself.

A knowledge of the theory is not necessary to solve equations. I did solve several equations and found the solutions to be very interesting, with the exception of the trinomial equation my opinion is the "Root Square Method" is the most practical.

Iteration Method

This process may be applied to obtain a more exact complex root when an approximate root has been found. If in solving by MacLaurin's series or Mc Clintock's method a slowly converging series is found. This method may be used to determine a root which is more exact.

Given the function

$$g(x) \equiv a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

If we have an approximate complex root e we may form the equation

$m(x) \equiv [x - (a + bi)][x - (a - bi)] = 0$, where $e = a + bi$,

since complex roots occur in pairs or

$m(x) \equiv x^2 - 2ax + (a^2 + b^2) = 0.$

then by the division algorithm

(1) $g(x) \equiv m(x)\,q(x) + r(x)$ where $r(x)$ is linear or zero

now the substitution of either e or its conjugate in (1) gives

$g(x) \equiv r(x)$, then $r(e)$ gives the value of $g(e)$.

In like manner the value of $g'(x)$ may be evaluated for $x = e$. $\quad g'(x) \equiv m(x)\,g'(x) + r'(x) \qquad g'(e) = r'(e)$

then a more exact root f will be, $f = e - \dfrac{g(e)}{g'(e)}$,

Obviously this method may be repeated to give as exact a root as desired. In solving $x^4 + 7x - 6 = 0$ by McClintock's method the approximate complex roots are found to

be $.6 + 1.7\,i$

now form the equation

$$m(x) \equiv [x - (.6 + 1.7\,i)][x - (.6 - 1.7\,i)] = 0.$$

$$m(x) \equiv x^2 - 1.2x + 2.89 = 0$$

now if $g(x) \equiv x^4 + 7x - 6 = 0$ then dividing $\frac{g(x)}{m(x)}$, and

finding the remainder, $r(x) = 1.792x - 1.81$

we have

$g(e) = r(e)$ or

$$g(.6 + 1.7\,i) = 1.792x - 1.81$$

$$= -.735 + 3.046\,i$$

now $g'(x) \equiv 4x^3 + 7 = 0$

and $\frac{g'(x)}{m(x)}$ gives a remainder $r'(x) = -5.80x - 6.872$

then $g'\,(.6 + 1.7\,i) = -5.80x - 6.872$

$$= -10.352 - 9.86\,i$$

now for a nearer approximation f

$$f = e - \frac{g(e)}{g'(e)}$$

$$f = (.6 + 1.7\,i) - \frac{-.735 + 3.046\,i}{-10.352 - 9.86\,i}$$

$$f = (.6 + 1.7\,i) - (.183 - .119\,i)$$

$f = .417 + 1.819\,i$ which is a more approximate value of the root.

The real roots were found to be $.800$, -2.159

Check. $r_1\,r_2\,r_3\,r_4 = -6.017$

SOLUTION of DE MOIVRES QUINTIC

Given the equation

(I) $x^5 - 5q\,x^3 + 5q^2\,x - c = 0$

Let $x = y + q/y$ then

$(y + q/y)^5 - 5q\,(y + q/y)^3 + 5q^2\,(y + q/y) - c = 0$

expanding and combining terms gives

$y^5 + \frac{q^5}{y^5} - c = 0$

$(y^5)^2 - e\,y^5 + q^5 = 0$

Solving as a quadratic

(II) $y^5 = \frac{c}{2} \pm \sqrt{\frac{c^2}{4} - q^5}$

then $y_1 = \sqrt[5]{\frac{c}{2} + \sqrt{\frac{c^2}{4} - q^5}}$, $\quad y_2 = \sqrt[5]{\frac{c}{2} - \sqrt{\frac{c^2}{4} - q^5}}$

then $y_1\,y_2 = q$ and if e is a fifth root of unity the ten values of y can be separated into pairs $y_1\,e^r$, $y_2\,e^{5-r}$ $(r = 0, 1, 2, ---4)$ such that the product of the two in a pair is $y_1\,y_2 = q$.

Now $x = y + q/y$ and the five roots of (I) are

(III) $x = y_1\,e^r + y_2\,e^{5-r}$ $(r = 0, 1, 2, ---4)$

Other forms of equations which can be solved by this method are

$x^4 - 4q\,x^2 + 2q^2 x - c = 0$

$x^7 - 7q\,y^5 + 14\,q^2\,y^3 - 7q^3 y - c = 0$

Example:

$$x^5 + 10x^3 + 20x + 31 = 0$$

$$-5q = 10, \quad 5q^2 = 20 - c = 31$$

$$q = 2, \quad q = +2 \quad c = -31$$

$$y^5 = -\frac{31}{2} \pm \sqrt{\frac{961}{4} - (-32)}$$

$$y^5 = -\frac{31}{2} \pm \frac{33}{2}$$

$$y^5 = 1, -32$$

$$y = 1, -2$$

$$x = (e^r - 2e^{5-r}) \quad r = (0,1,2,----4)$$

where e is the fifth root of unity or

$$e^0 = 1, \; e^1 = .30902 + .95106i, \; e^2 = -.80902 + .58779i$$

$$e^3 = -.80902 - .58779i, \quad e^4 = .30902 - .95106i$$

when $r = 0$ $\quad x_0 = -1$

$r = 1$ $\quad x_1 = -.30902 + 2.85318i$

$r = 2$ $\quad x_2 = .80902 + 1.76337i$

$r = 3$ $\quad x_3 = .80902 - 1.76337i$

$r = 4$ $\quad x_4 = -.30902 - 2.85318i$

check $x_0 \, x_1 \, x_2 \, x_3 \, x_4 = -31.0005$

SUMMARY

Root Square Method:

I believe this to be the best method if all the roots of the equation are to be determined. The only time difficulty might be encountered is when two or more of the absolute values of the real roots, or moduli of the complex roots, should happen to be quite near together. In this case the process might be very protracted and one of the other methods would be advisable.

The use of a table of squares will facilitate the work. Great care should be taken to make no mistakes in signs or products, since an error of this kind might easily lead to almost as much work as the original solution, and the solution is laborius enough in itself.

If several of the terms are missing in the equation then a solution by MacLaurin's Series or the method of McClintock might be advisable. But in this case a large number of the products in the Root Square Method would be zero for the first powers of the roots.

Solution by MacLaurin's Series

If several of the terms of the equation are missing it might be advisable to try this method, and see if the series will converge quite rapidly. This can be done quite easily and the advantage gained if it should be a

rapidly converging series, will amply repay the work of the testing. In any other case my objection to this method is the work of the differentiations as they may become very involved. Also the calculation of the numerical values of these differentials.

The possibilities of linear transformations to make rapidly converging series should not be overlooked.

By this method the complex roots may be determined without regard to the real roots. In certain cases it might be advisable to do this and find the real roots by Horner's method, Newton-Raphson, or some other method if all roots are desired.

Iteration Method:

When under any method an approximate complex root has been found, the root may be found to any degree of accuracy by the repeated application of this method. If only complex roots are desired and the series under MacLaurin's Series are slowly convergent, then this method is very satisfactory. The proof that Newton's Method of approximation holds for complex roots as well as real roots may be found in "Theory of Equations--Todhunter." Pgs. 133-134.

DeMoivre's Quintic and Analogous Forms:

If the equation can be put in one of these forms a quick and exact solution can be made. Especially if the various roots of one have been predetermined or are avaliable.

General:

The theorems for the determination of real roots should be kept constantly in mind. In many cases it may be practical to use these to determine the real roots in conjunction with the methods for determining the complex roots.

BIBLIOGRAPHY

The Solution of Equations--Mansfield Merriman

Pages 10-12, 23-24, 28-37.

Elementary Theory of Equations--Dickson

Pages 75-76

Theory of Equations--Todhunter

Pages 133-134

The Calculus of Observations--Whittaker & Robinson

Pages 106-118

Numerical Mathematical Analysis--Scarborough

Pages 198-217

Practical Mathematical Analysis--H. Von Sanden

Pages 51-54

American Journal of Mathematics, 1895, Vol. XVII.

A method for Calculating Simultaneously all the Roots of an Equation. By Emory McClintock

Pages 90-130

Theory of Equations--Burnside & Panton